Homogeneous Cosmologies in Generalized Modified Gravity

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Abstract Dynamical system methods are used in the study of the stability of spatially flat homogeneous cosmologies within a large class of generalized modified gravity models in the presence of a relativistic matter-radiation fluid. The present approach can be considered as the generalization of previous works in which only $F(R)$ cases were considered. Models described by an arbitrary function of all possible geometric invariants are investigated and general equations giving all critical points are derived.

Keywords Modified gravity models · Dark energy · Stability

1 Introduction

Recent cosmological data support the fact that there is a strong evidence for a late accelerated expansion of the observable universe, apparently due to the presence of an effective positive and small cosmological constant of unknown origin. This is known as dark energy issue (see for example [1–5]).

Modified gravity models are possible realizations of dark energy (for a recent review and alternative approaches see [6, 7]), which may offer a quite natural geometrical approach again in the spirit of the original Einstein theory of gravitation. In fact, the main idea underlying these approaches to dark energy puzzle is quite simple and consists in adding to the gravitational Einstein-Hilbert action other gravitational terms which may dominate the cosmological evolution during the very early or the very late universe epochs, but in such a way that General Relativity remains valid at intermediate epochs and also at non cosmological scales. They are generalization of the ΛCDM model, namely Einstein gravity plus a

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positive cosmological constant, the simplest model for dark energy, which however has to be confronted with well known difficulties, among them, the cosmological constant problem, an unsolved issue so far. We also note that realistic models of f(R) gravity may realistically describe not only unification of inflation with dark energy but also dark matter as it was demonstrated in [8].

In the present paper, first of all we shall generalise the analysis presented in [9], where the stability of the de Sitter solution (the vacuum invariant submanifold) has been investigated in a class of modified gravitational cosmological models defined in a Friedmann-Robertson-Walker (FRW) spatially flat space-time. The method is well known and consists in rewriting the equations of motion as a system of first-order autonomous differential equations and makes use of the theory of dynamical systems (see [10–16] and references therein). We remind that the stability or instability issue is really relevant in cosmology. For example, in the ΛCDM model it ensures that no future singularities will be present in the solution. Within cosmological models, the stability or instability around a solution is of interest at early and also at late times.

Then we will introduce also matter/radiation in the model and we shall generalise the method given in [12], which permits to determine all critical points of a $F(R)$ model. Ordinary matter is important in reconstructing the expansion history of the Universe and probing the phenomenological relevance of the models (see for example the recent papers [12–16], where the $F(R)$ case has been discussed in detail). Our generalisation consists in the extension of that method in order to include all possible geometrical invariants. This means that F could be a generic scalar function of curvature, Ricci and Riemann tensors.

To our knowledge, besides the paper [17], the dynamical system analysis have been used and critical points derived and analysed mainly for models described by an action of the kind

$$ S = S_m + \frac{1}{2\chi} \int d^4x \sqrt{-g} F(R), \quad \chi = 8\pi G_N, \tag{1.1} $$

$F(R)$ being an arbitrary function of the scalar curvature R, G_N the Newton's constant and S_m the matter action. The simplest choice $F(R) = R - 2\Lambda$ corresponds to the ΛCDM model. Here we shall propose a method which will permit to derive the critical points also for models described by a function $F(U_k)$, U_k being generic geometric (higher-order) invariants (see the Appendix).

Their interest has been recently triggered by the appearance of [18–20] and after that investigated in many papers (see, for example [21–47]). Furthermore, very recently in [48–55], new "viable" models have been introduced and discussed. In particular, in [56] the possibility that also dark matter effects may be described within this class of models is suggested.

With regard to the stability of the solutions, it has been discussed in several places, an incomplete list being [57–71] and earlier references quoted in [49]. To this aim, different techniques have been employed, including manifestly covariant and field theoretical approaches, where the gauge issue has been properly taken into account. All these investigations are in agreement with the following conditions which ensures the existence and the stability of the de Sitter solution with scalar curvature $R = R_0$:

$$ \begin{cases} 2F(R_0) - R_0 F'(R_0) = 0, & \text{existence,} \\ \frac{F'(R_0)}{R_0 F''(R_0)} - 1 > 0, & \text{stability,} \end{cases} \tag{1.2} $$

where F' and F'' are the derivatives of $F(R)$ with respect to R and everywhere in the paper the subscript 0 indicates quantities evaluated at the critical point. The first condition in (1.2)

determines the scalar curvature of the de Sitter solution, while the second one gives the condition for the stability around such a solution.

There are some theoretical (quantum effects and string-inspired) motivations in order to investigate gravitational models depending on higher-order invariants. The "string-inspired" scalar-Gauss-Bonnet gravity case $F(R, G)$ has been suggested in [72] as a model for gravitational dark energy, while some time ago it has been proposed as a possible solution of the initial singularity problem [73–75]. The investigation of different regimes of cosmic acceleration in such gravity models has been carried out in [72, 76–93]. In particular, in [89] a first attempt to the study of the stability of such kind of models has been carried out using an approach based on quantum field theory.

The method we shall use in the present paper is based on a classical Lagrangian formalism [94–96], inspired by the seminal paper [97], where quantum gravitational effects were considered for the first time. With regard to this, it is well known that one-loop and two-loops quantum effects induce higher derivative gravitational terms in the effective gravitational Lagrangian. Instability due to quadratic terms have been investigated in [98–100]. A particular case has been recently studied in [101–103] and general models depending on quadratic invariants have been investigated in [104–107].

A stability analysis of nontrivial vacua in a general class of higher-derivative theories of gravitation has already been presented in [108]. Our approach is different from the one presented there since we are dealing with scalar quantities and moreover it is more general, since it is not restricted to the vacuum invariant submanifold.

Finally, it should be stressed that the stability studied here is the one with respect to homogeneous perturbations. For the $F(R)$ case, the stability criterion for homogeneous perturbations is equivalent to the inhomogeneous one [58–60].

The content of the paper is the following. In Sect. 2 we describe our general method in the vacuum case and derive general conditions for existence and stability of de Sitter solutions for arbitrary $F(U_k)$ models. Then, in Sect. 3, we extend the method in order to include matter/radiation and give the conditions which (in principle) permit to derive all critical points. In Sect. 4 we analyse some specific models and finally we make some conclusions in Sect. 5. For the reader convenience, the paper ends with an Appendix in which we derive some interesting useful relations.

2 The General Case without Matter

In this section we would like to study the stability condition of de Sitter solutions for models in which the Lagrangian density is an arbitrary function of all algebraic invariants built up with the Riemann tensor of the FRW space-time we are dealing with, that is

$$\mathcal{L} = -\frac{1}{2\chi} F(R, P, Q, \ldots), \tag{2.1}$$

where R is the scalar curvature, $P = R^{\mu\nu} R_{\mu\nu}$ and $Q = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ are the two quadratic invariants and the dots means other independent algebraic invariants of higher order.

For the sake of convenience we write the metric in the form

$$ds^2 = -e^{2n(t)}dt^2 + e^{2\alpha(t)}d\vec{x}^2, \qquad N(t) = e^{n(t)}, \qquad a(t) = e^{\alpha(t)}. \tag{2.2}$$

In this way $\dot{\alpha}(t) = H(t)$ is the Hubble parameter and the generic invariant quantity U has the form

$$U = e^{-2pn(t)} u(\dot{n}, \dot{\alpha}, \ddot{\alpha}) = e^{-2pn(t)} u(\dot{n}, H, \dot{H}) = H^{2p} e^{-2pn(t)} u(X), \tag{2.3}$$

where $2p$ is the dimension (in mass) of the invariant under consideration and $X = (\dot{H}/H^2 - \dot{n}/H)$ (see the Appendix). In particular one has

$$R = 6e^{-2n}\left[2H^2 + \dot{H} - \dot{n}H\right] = 6H^2e^{-2n}(2 + X),$$

$$P = 12e^{-4n}\left(\dot{n}^2H^2 - 3\dot{n}H^3 - 2\dot{n}H\dot{H} + 3H^4 + 3H^2\dot{H} + \dot{H}^2\right) = 12H^4e^{-4n}(3 + 3X + X^2),$$

$$Q = 12e^{-4n}\left(\dot{n}^2H^2 - 2\dot{n}H^3 - 2\dot{n}H\dot{H} + 2H^4 + 2H^2\dot{H} + \dot{H}^2\right) = 12H^4e^{-4n}(2 + 2X + X^2),$$

$$G = R^2 - 4P + Q = 24e^{-4n}(H^4 + H^2\dot{H}) = 24H^4e^{-4n}(1 + X). \tag{2.4}$$

Using this notation, the action reads

$$S = -\int d^3x \int dt\, L(n, \dot{n}, \alpha, \dot{\alpha}, \ddot{\alpha}) = \frac{1}{2\chi} \int d^3x \int dt\, e^{n+3\alpha} F(n, \dot{n}, \alpha, \ddot{\alpha}) \tag{2.5}$$

and the Lagrange equations corresponding to the two Lagrangian variables $n(t)$ and $\alpha(t)$ are given by

$$E_n = \frac{\partial L}{\partial n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{n}} = 0, \tag{2.6}$$

$$E_\alpha = \frac{\partial L}{\partial \alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{\alpha}} = 0. \tag{2.7}$$

Since the function $n(t)$ is a "gauge function", which corresponds to the choice of the evolution parameter, we expect (2.7) to be a direct consequence of (2.6) and in fact a straightforward calculation leads to the identity (see (A.8) in the Appendix)

$$\frac{dE_n}{dt} = \dot{n}E_n + \dot{\alpha}E_\alpha. \tag{2.8}$$

As a consequence of such an identity we may limit our analysis to the use of (2.6), much simpler than (2.7). Furthermore, we may use the gauge freedom and fix the cosmological time by means of the condition $N(t) = 1$, that is $n(t) = 0$. From now on it is understood that all quantities will be evaluated in such a gauge and so the parameter t corresponds to the standard cosmological time. In this way (2.6) and (2.7) read

$$F + F_n - 3HF_{\dot{n}} - \dot{F}_{\dot{n}} = H\dot{F}_{\dot{H}} - HF_H + F - \dot{H}F_{\dot{H}} + 3H^2F_{\dot{H}} = 0, \tag{2.9}$$

$$\ddot{F}_{\dot{H}} - \dot{F}_H + 6H\dot{F}_{\dot{H}} - 3HF_H + 3F + 3\dot{H}F_{\dot{H}} + 9H^2F_{\dot{H}} = 0, \tag{2.10}$$

where $F_X = \partial_X F$ means derivative of F with respect to X. All derivatives with respect to n in (2.9) have been eliminated by using (A.6) in the Appendix.

In the next section we shall study all critical points of the system in the presence of matter, while here we shall limit ourselves to determine the conditions for the existence and the stability of de Sitter solutions in the absence of matter. To this aim we take into account (2.9) and re-write it as a differential equation for the Hubble parameter $H(t)$, that is

$$HF_{\dot{H}\dot{H}}\ddot{H} + H(F_{H\dot{H}} - F_{\dot{H}})\dot{H} + \Sigma(H, \dot{H}) = 0, \tag{2.11}$$

where we have set

$$\Sigma(H, \dot{H}) = F - HF_H + 3H^2F_{\dot{H}}. \tag{2.12}$$

We see that the condition for the existence of a de Sitter solution $P_0 \equiv (\dot{H} = 0, H = H_0)$ is

$$\Sigma(H_0, 0) = 0 \quad \Longrightarrow \quad \left[F - HF_H + 3H^2 F_{\dot{H}}\right]_{P_0} = 0, \tag{2.13}$$

where here all quantities have to be evaluated at P_0. The latter condition, which gives rise to the de Sitter critical point, can be simplified by means of (A.9) in the Appendix. In fact we have

$$\left[F - HF_H + 3H^2 F_{\dot{H}}\right]_{P_0} = \left[F - H^2 F_{\dot{H}}\right]_{P_0} = 0. \tag{2.14}$$

In order to study the stability of such a solution we have to distinguish between two possible cases. The simplest one occurs when F is linear in $\dot{H}$, that is $F_{\dot{H}\dot{H}} = 0$. In such a case (2.11) assumes the form

$$(HF_{H\dot{H}} - HF_{\dot{H}} - \Sigma_{\dot{H}})\dot{H} + \Sigma(H, 0) = \Sigma(H, 0) = 0 \quad \Longrightarrow \quad H = H_0. \tag{2.15}$$

Then we see that in such a special case the solution, if it exists, is trivially stable, since the field equation reduces to an algebraic equation which fixes the value of H in terms of the parameters of the system. The Einstein's theory with cosmological constant $F = R - 2\Lambda$ is an example of this kind and in fact (2.15) gives rise to $H^2 = H_0^2 = \Lambda/3$.

In the second case $F_{\dot{H}\dot{H}} \neq 0$, in order to study the stability one may transform (2.11) in an autonomous system by introducing the functions $K(t) = \dot{H}(t)$. In this way, one gets

$$\dot{H} = K, \qquad \dot{K} = -\frac{\Sigma(H, K) + (HF_{H\dot{H}} - F_{\dot{H}})K}{HF_{\dot{H}\dot{H}}}. \tag{2.16}$$

The critical points are determined by the condition $(\dot{H} = 0, \dot{K} = 0)$ and the linearized system which determines the stability reads

$$\begin{pmatrix} \delta\dot{H} \\ \delta\dot{K} \end{pmatrix} = M \begin{pmatrix} \delta H \\ \delta K \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ A_0 & B_0 \end{pmatrix}, \tag{2.17}$$

where

$$A_0 = \left[-\frac{\Sigma_H}{HF_{\dot{H}\dot{H}}}\right]_{P_0} = \left[\frac{F_{HH} - 3HF_{H\dot{H}} - 6F_{\dot{H}}}{F_{\dot{H}\dot{H}}}\right]_{P_0}, \tag{2.18}$$

$$B_0 = \left[\frac{F_{\dot{H}} - HF_{H\dot{H}} - \Sigma_{\dot{H}}}{HF_{\dot{H}\dot{H}}}\right]_{P_0} = -H_0. \tag{2.19}$$

The stability conditions are obtained by requiring $\operatorname{Tr} M < 0$ and $\det M > 0$. The first condition is trivially satisfied since H_0 is positive, while the second one gives

$$\left[\frac{3HF_{H\dot{H}} - F_{HH} + 6F_{\dot{H}}}{F_{\dot{H}\dot{H}}}\right]_{P_0} > 0. \tag{2.20}$$

Summarizing, we have the general results

$$\begin{cases} \left[F - H^2 F_{\dot{H}}\right]_{P_0} = 0, & \text{critical points,} \\ \left[\frac{3HF_{H\dot{H}} - F_{HH} + 6F_{\dot{H}}}{F_{\dot{H}\dot{H}}}\right]_{P_0} > 0, & \text{stability condition when } F_{\dot{H}\dot{H}} \neq 0. \end{cases} \tag{2.21}$$

In the particular case in which $F_{\dot{H}\dot{H}} = 0$ the eventual (de Sitter) critical point is always stable. Of course, when $F = F(R)$, (2.21) become equivalent to (1.2).

3 The General Case with Matter

The inclusion of matter in the model is obtained by adding the matter action S_m to the gravitational action (2.5). Of course, in order to preserve symmetry we have to consider homogeneous and isotropic matter, that is a perfect fluid. First of all we observe that

$$\frac{\delta S_m}{\delta n}\delta n = -\int d^4x\sqrt{-g}\,T_{00}g^{00}\delta n, \tag{3.1}$$

$$\frac{\delta S_m}{\delta \alpha}\delta \alpha = -\int d^4x\sqrt{-g}\,T_{ab}g^{ab}\delta \alpha, \tag{3.2}$$

where $T_{\mu\nu} \equiv (T_{00}, T_{ab})$ is the energy-momentum tensor, which for a perfect fluid satisfies the conditions

$$T_{00}g^{00} = -\rho, \qquad T_{ab}g^{ab} = 3p, \quad a,b = 1,2,3,$$

ρ and $p = p(\rho)$ being respectively the density and the pressure of matter.

Now (2.6) and (2.7) trivially changes for the presence of matter and in fact we get

$$E_n = \frac{\partial L}{\partial n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{n}} = 2\sqrt{-g}\,T_{00}g^{00} = 2\rho\sqrt{-g}, \tag{3.3}$$

$$E_\alpha = \frac{\partial L}{\partial \alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{\alpha}} = \sqrt{-g}\,T_{ab}g^{ab} = -6p\sqrt{-g}. \tag{3.4}$$

The identity (2.8) is still valid and is equivalent to the energy-momentum conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \quad\Longrightarrow\quad \dot{\rho} = -3H(\rho + p). \tag{3.5}$$

Using the notation of the previous section and putting again $n(t) = 0$ now we get

$$H\dot{F}_{\dot{H}} - HF_H + F - \dot{H}F_{\dot{H}} + 3H^2F_{\dot{H}} = 2\rho, \tag{3.6}$$

$$\ddot{F}_{\dot{H}} - \dot{F}_H + 6H\dot{F}_{\dot{H}} - 3HF_H + 3F + 3\dot{H}F_{\dot{H}} + 9H^2F_{\dot{H}} = -6p. \tag{3.7}$$

The latter equations are the generalisation to arbitrary action of the well known Friedmann equations. It is interesting to observe that in the pure Einstein gravity, that is for $F = R$, they read

$$H^2F_{\dot{H}} = F_X = 2\rho \quad\Longrightarrow\quad \Omega_\rho = 1, \tag{3.8}$$

$$(3H^2 + 2\dot{H})F_{\dot{H}} = (3 + 2X)F_X = -6p \quad\Longrightarrow\quad \Omega_p = -1 - \frac{2}{3}X, \tag{3.9}$$

where we have introduced the dimensionless variables

$$\Omega_\rho = \frac{2\rho}{H^2F_{\dot{H}}} = \frac{2\rho}{F_X}, \qquad \Omega_p = \frac{2p}{H^2F_{\dot{H}}} = \frac{2p}{F_X}, \tag{3.10}$$

which in this special case are given by the usual values $\Omega_\rho = \rho/3H^2$ and $\Omega_p = p/3H^2$. From (3.8) and (3.9) it follows

$$w \equiv \frac{p}{\rho} = \frac{\Omega_\rho}{\Omega_p} = -1 - \frac{2}{3}X. \tag{3.11}$$

In the general case, (3.6) and (3.7) have more terms with respect to (3.8) and (3.9) and it is quite natural to interpret them as corrections due to the presence of higher-order terms in the action. Then we define

$$\Omega_\rho^{eff} = \Omega_\rho + \Omega_\rho^{curv} = 1, \qquad \Omega_p^{eff} = \Omega_p + \Omega_p^{curv} = -1 - \frac{2}{3}X, \qquad (3.12)$$

$$w_{eff} \equiv \frac{\Omega_p^{eff}}{\Omega_\rho^{eff}} = -1 - \frac{2}{3}X, \qquad (3.13)$$

where Ω_ρ^{curv} and Ω_p^{curv} are complicated expressions, which only depend on the function F. They can be derived from (3.6) and (3.7), but they explicit form is not necessary for our aims. The effective quantity w_{eff} is equal to the ratio between the effective density and the effective pressure and it could be negative even if one considers only ordinary matter. It is known that the current-measured value of w_{eff} is near -1.

In order to get all critical points of the system now we follow the method described in [12]. First of all we introduce the dimensionless variables

$$\Omega_\rho = \frac{2\rho}{H^2 F_{\dot H}} = \frac{2\rho}{F_X}, \qquad \Omega_p = \frac{2p}{H^2 F_{\dot H}} = \frac{2p}{F_X}, \qquad (3.14)$$

$$X = \frac{\dot H}{H^2}, \qquad Y = \frac{F - H F_H}{H^2 F_{\dot H}} = \frac{F}{F_X} - X, \qquad Z = \frac{\dot F_{\dot H} - F_H}{H F_{\dot H}} = \frac{F_X'}{F_X} - 2X - \xi, \quad (3.15)$$

where the prime means derivative with respect to α and the quantity

$$\xi = \xi(X, Y) = \frac{F_H}{H F_{\dot H}} = \frac{H F_H}{F_X}, \qquad (3.16)$$

has to be considered as a function of the variables X and Y. In general it is a function of X and H, but this latter quantity can be expressed in terms of X and Y as a direct consequence of the definition of Y itself. Then we derive an autonomous system by taking the derivatives of such variables. From (3.6) we have the constraint

$$\Omega_\rho = Y + Z + 3 \implies \Omega_{curv} = -(Y + Z + 2), \qquad (3.17)$$

which reduces to the standard one when F is linear in R (general relativity with cosmological constant).

Differentiating the variables above by taking into account of (3.5) and (3.7) we get the system of first order differential equations

$$\begin{cases} X' = -2X^2 - \gamma X + \beta(Z + \xi), \\ Y' = -(2X + Z + \xi)Y - XZ, \\ Z' = -3\Omega_p - [X(Z+6) + Z(Z + \xi + 6) + 3(Y + \xi + 3)], \\ \Omega_\rho' = -3(\Omega_\rho + \Omega_p) - (2X + Z + \xi)\Omega_\rho, \\ \Omega_p' = -3\frac{dp}{d\rho}(\Omega_\rho + \Omega_p) - (2X + Z + \xi)\Omega_p, \end{cases} \qquad (3.18)$$

where $X' \equiv \frac{dX}{d\alpha} = \frac{1}{H}\frac{dX}{dt}$ and so no and $p = p(\rho)$ has been assumed. The first equation in (3.18) has been obtained by deriving the function $F_{\dot H}$ with respect to t and putting

$$\beta = \beta(X, Y) = \frac{F_{\dot H}}{H^2 F_{\dot H \dot H}} = \frac{F_X}{F_{XX}}, \qquad \gamma = \gamma(X, Y) = \frac{F_{H \dot H}}{H F_{\dot H \dot H}} = \frac{H F_{HX}}{F_{XX}} = \beta \xi_X + \xi. \qquad (3.19)$$

It is understood that $F_{\dot{H}\dot{H}} \neq 0$ has been assumed.

Note that in principle one could consider a mixture of different kind of matter/radiation with densities ρ_k and corresponding pressures p_k. In such a case in (3.18) there will appear the sums of the corresponding quantities Ω_{ρ_k} and Ω_{p_k} and the corresponding differential equations for any type of matter/radiation. For simplicity here we consider only one type of matter/radiation described by the equation of state $p = p(\rho)$.

In general, among the five differential equations (3.18) only three of them are linear independent for the presence of the constraint (3.17) and the equation of state. For simplicity now we assume the pressure to be proportional to the density, that is $p = w\rho$, with constant w, where for ordinary matter $0 \leq w \leq 1/3$ ($w = 0$ corresponds to dust, while $w = 1/3$ to pure radiation), but in principle one could also consider "exotic" matter with $w < 0$ and cosmological constant which corresponds to $w = -1$. With this choice and from (3.17) we get

$$\Omega_p = w\Omega_\rho, \qquad \Omega_\rho = Z + Y + 3. \tag{3.20}$$

As a consequence the autonomous system which gives rise to the critical points can be chosen as

$$\begin{cases} 0 = X' = -2X^2 - \gamma X + \beta(Z + \xi) \\ 0 = Y' = -(2X + Z + \xi)Y - XZ \\ 0 = Z' = -3(1 + w)(Z + Y + 3) - (Z + \xi)(Z + 3) - X(Z + 6) \end{cases} \tag{3.21}$$

and Ω_ρ at the critical points will be determined by means of (3.17). The critical points are the solutions of the algebraic system (3.21). The number and the position of such points depends on the Lagrangian throughout the functions β, γ and ξ. In principle, given F one can derive all critical points as the constant solutions of (3.21), but in practice for a generic F the algebraic system could be very complicated and the solutions quite involved. We shall consider in detail some particular cases in the next section.

A brief comment about the equivalence of (3.21) and (3.6)–(3.7) is in order. It is well known that the Gauss-Bonnet scalar G does not contribute to the field equations, because it is a topological invariant and this means that the field equations (3.6)–(3.7) are invariant with respect to the transformation $F \rightarrow F + cG$, c being an arbitrary constant. On the contrary, the new variables X, Y, Z (by definitions) are not invariant with respect to such a transformation and so the system (3.21) could contain the parameter γ explicitly, but nevertheless the solutions for $H(t)$ do not depend on such a parameter. The same thing is true for the critical points.

It has also to be noted that in general, given a Lagrangian F, there are solutions of the system (3.21), which give rise to trivial or unphysical models. Of course such solutions have to be dropped (see specific examples).

Before to analyse all possible solutions of (3.21), we observe that in the absence of matter the de Sitter solution $X = 0$ of course is a critical point for the system if the first equation in (2.21) is satisfied. In fact we trivially see that $X = 0$, $\Omega_\rho = Y + Z + 3 = 0$ satisfy (3.21) if $Y - \xi + 3 = 0$, which is equivalent to first equation in (2.21) when $X = 0$. (Strictly speaking, the condition $Y = 1$ is equivalent to (2.21), when $H_0 \neq 0$. In principle, (2.21) can also have the Minkowskian solution $R_0 = 0$.) On the de Sitter solution the value of w_{eff} is exactly equal to -1. By taking the variations of (3.21) with respect to X and Y on the critical point and using the identity (A.9) in the Appendix we also get the stability condition in (2.21).

In order to show that the present approach is a generalisation of the one proposed in [12], we observe that when the function $F = F(R)$ depends on the unique invariant $U = R$, we

exactly obtain all known results. For such a special case, using (2.4) (with $n(t) = 0$) and (A.5)–(A.6) in the Appendix we get

$$\xi = \frac{H F_H}{F_X} = \frac{2pU}{U_X} - 2X = 4, \tag{3.22}$$

$$\beta = \frac{F_X}{F_{XX}} = \frac{F_U U_X}{F_U U_{XX} + F_{UU} U_X^2} = \frac{F_R}{6H^2 F_{RR}}, \tag{3.23}$$

$$\gamma = \frac{H F_{HX}}{F_{XX}} = \beta \xi_X + \xi = 4. \tag{3.24}$$

Now with the trivial replacement

$$X = x_3 - 2, \qquad Y = x_2 - x_3 + 2, \qquad Z = -(x_1 + 4), \qquad \beta = \frac{x_3}{m}. \tag{3.25}$$

Equations (3.18) become equivalent to the analog equations in [12], where this class of models have been studied in detail.

As already said above, the critical points of (3.21) depends on ξ, β, γ, which in general are complicated functions of X and Y, then it is not possible to determine general solutions without to choose the model, nevertheless it is convenient to distinguish two distinct classes of solutions characterised by the values of $w \neq -1$ and $w = -1$. For the sake of completeness we consider $w \leq 1/3$ and so we write the solutions also for "exotic" matter, that is quintessence ($-1 < w < 0$) and phantom ($w < -1$). Of course, such solutions have to be dropped if one is only interested in ordinary matter/radiation. We have

- $w \neq -1$—The critical points are the solutions of the system of three equations

$$\begin{cases} 2X^2 + \gamma X - \beta(Z + \xi) = 0 \\ (2X + Z + \xi)Y + XZ = 0 & w \neq -1, \quad (3.26) \\ 3(1 + w)(Z + Y + 3) + (Z + \xi)(Z + 3) + X(Z + 6) = 0 \end{cases}$$

where ξ, β, γ are functions of X, Y determined by (3.16) and (3.19). The stability matrix has three eigenvalues and the point is stable if the real parts of all of them are negative.

The latter system has always the de Sitter solution $P_0 \equiv (X = 0, Y = 1, Z = -4)$, where $\Omega_\rho = 0$ and $w_{eff} = -1$. Note however that such a solution could exist also in the presence of matter, since the existence of P_0 critical point only implies that the critical value for Ω_ρ vanishes.

- $\Omega_\rho \neq 0$, $w = -1$—The critical points are given by

$$\begin{cases} 2X^2 + \gamma X - \beta(Z + \xi) = 0 \\ (2X + Z + \xi)Y + XZ = 0 & w = -1. \quad (3.27) \\ (Z + \xi)(Z + 3) + X(Z + 6) = 0 \end{cases}$$

For this class of solutions, the non-singular stability matrix has three eigenvalues and the point is stable if the real parts of all of them are negative.

We see that there is at least one singular case (critical line) when $X = 0$ and $Z = -\xi = -4$. In fact in such a case Y or Ω_ρ are undetermined since

$$\Omega_\rho = Y + 3 - \xi(0, Y) = Y - 1 \implies Y = 1 + \Omega_\rho, \quad \Omega_\rho \text{ arbitrary.} \tag{3.28}$$

Such a solution can be seen as a generalisation of the de Sitter solution for a model with cosmological constant. The de Sitter critical point for the model $\tilde{F} = F - 2\Lambda$ reads $(X = 0, \tilde{Y} = 1, \tilde{Z} = -4)$. Such a solution follows from (3.28) if we choose $\rho_0 \equiv \Lambda$. In fact, on the critical point $(X = 0, Y = 1 + \Omega_\rho, Z = -4)$ (3.28) and from definitions (3.15) we get

$$\Omega_\rho = \frac{2\rho}{H^2 F_{\dot{H}}} = \frac{F}{H^2 F_{\dot{H}}} - 1 \quad \Longrightarrow \quad \tilde{Y} \equiv \frac{\tilde{F}}{H^2 \tilde{F}_{\dot{H}}} = 1, \tag{3.29}$$

which corresponds to de Sitter critical point for $\tilde{F}$. Of course, (3.28) is more general than the case with pure cosmological constant since ρ is not necessary a constant.

Of course for this special class of solutions $w_{eff} = -1$. Note also that the stability matrix has always a vanishing eigenvalue and the stability of the system is determined by the other two eigenvalues.

For some models, but just for technical reasons, it could be convenient to treat the cosmological constant as matter, using the previous identification we have done.

4 Explicit Examples

In order to see how the method works, now we give explicit solutions for some models and, when possible we also study the stability of the critical points. We restrict our analysis to the values $0 \leq w \leq 1/3$ and to the special value $w = -1$, which corresponds to the pure cosmological constant, but in principle any negative value of w could be considered, even if this will be in contrast with the aim of modified gravity. In fact, modified gravity can generate an effective negative value of w without the use of phantom or quintessence.

It as to be stressed that in general, due to technical difficulties, one has to study the models by a numerical analysis. Only for some special cases one is able to find analytical results. Here we report the results for some models of the latter class in which the analytical analysis can be completely carried out. We also study more complicated models and for those we limit our analysis to the de Sitter solutions. The phenomenological analysis of such models can be found in [109].

In the following we shall use the compact notation

$$P \equiv (X, Y, Z, \Omega_\rho, w_{eff}), \qquad P_0 \equiv (0, 1, -4, 0, -1),$$
$$P_\Lambda = (0, 1 + \Omega_\Lambda, -4, \Omega_\Lambda, -1). \tag{4.1}$$

The latter is an additional critical point that we have for the choice $w = -1$ and can be seen as the de Sitter solution in the presence of cosmological constant.

$F = R - \mu^4/R$—This is the well known model introduced in [18–20] and discussed in [12]. For this model the system (3.21) with arbitrary w has six different solutions, but only two of them effectively correspond to physical critical points, if $0 \leq w \leq 1/3$. In principle there are other critical points for negative values of w (phantom or quintessence) and moreover there is also a particular solution for $w = -1$ which corresponds to the model with a cosmological constant Λ.

Solving the autonomous system one finds

- $P = P_0$: unstable de Sitter critical point. The critical value for the scalar curvature reads $R_0 = \sqrt{3}\mu^2$.
- $P = (-1/2, -1, -2, 0, -2/3)$: stable critical point. At the critical value, $H_0 = 0$.

- $P = (3(1 + w)/2, -(5 + 3w), -2(5 + 3w), -3(4 + 3w), -(2 + w))$: unstable critical point where $H_0 = 0$.
- $P = P_\Lambda$: unstable critical point. At the critical value one has $H_0^2 = (\Lambda/6)(1 + \sqrt{1 + 3\mu^4/4\Lambda^2}$.

$F = R + aR^2 + bP + cQ$—(Starobinsky-like model). Here we have to assume $3a + b + c \neq 0$ otherwise the quadratic term becomes proportional to the Gauss-Bonnet invariant. For $0 \leq w \leq 1/3$, this model has only one critical point. In order to have a de Sitter solution, we have to introduce a cosmological constant Λ. We have in fact

- $P = P_0$: Minkowskian solution with $R_0 = 0$, which is stable if $3a + b + c > 0$.
- $P = P_\Lambda$: de Sitter critical point with $R_0 = 6\Lambda$, which is stable if $3a + b + c > 0$, in agreement with [107].

$F = R - d^2Q_3$—This is the simplest toy model with the cubic invariant $Q_3 = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\mu\nu} R_{\gamma\delta}^{\mu\nu}$. For this model we have the following critical points:

- $P = P_0$: unstable de Sitter solution with $R_0 = 6/d$.
- $P = P_0$: stable Minkowskian solution with $R_0 = 0$.
- $P = (0.05, 0.60, -3.60, 0, -1.03)$: stable solution with $H_0 = 0$.
- $P = P_\Lambda$: this point exists, but not for any value of d and Λ. Also the value of H_0 and the stability depend on the parameters.

$F = R + aR^2 + bP + cQ - d^2Q_3$—This is a generalisation of the previous two models. It may be motivated by the two-loop corrections in quantum gravity [110, 111]. The de Sitter critical points have been studied in [9]. The algebraic equations (3.21) are too complicated to be solved analytically, but it is easy to verify that there are at least the following solutions:

- $P = P_0$: de Sitter solution with $R_0 = 6/d$. This is stable if $3a + b + c + 3d > 0$.
- $P = P_0$: Minkowskian solution with $R_0 = 0$, which is stable if $3a + b + c > 0$.
- $P = P_\Lambda$: also in this case this point exists and is stable depending on the parameters (see [9]).

5 Conclusion

In this paper we have presented a general technique which permits to arrive at a first order autonomous system of differential equations classically equivalent to the equations of motion for models of modified gravity based on an arbitrary function $F(R, P, Q, Q_3 \ldots)$, namely built up with all possible geometric invariant quantities of the FRW space-time. Dynamical system techniques have been applied to the investigation of critical points. We have shown that, in the special case of $F(R)$ theories, the method gives rise to the well known results [12–15], but in principle it can be applied to the study of much more general cases.

As applications, we have considered some simple models, for which a complete analytical analysis have been carried out. However, in general, due to technical difficulties, a numerical analysis is required. Among the models investigated, we would like to remind that we were able to deal with one which involves a cubic invariant in the curvature tensor and, to our knowledge, this has never been considered before, and this shows the power of our approach.

Appendix

Here we show that in the cases we are considering, (2.8) is always satisfied. To this aim, as in (2.3), we denote by $U(x)$ a generic invariant quantity of dimension $2p$ (in mass) computed in the system of coordinates $\{x\} \equiv \{(t, \vec{x})\}$ and by $\tilde{U}(y)$ the same quantity computed in the system $\{y\} \equiv \{(\tau, \vec{y})\}$. Such coordinates are chosen in such a way that $d\tau = e^{n(t)}dt$ and as a consequence the metrics read

$$ds^2 = -e^{2n(t)}dt^2 + e^{2\alpha(t)}d\vec{x}^2 = -d\tau^2 + e^{2\tilde{\alpha}(\tau)}d\vec{y}^2. \tag{A.1}$$

Recalling that U is built up with the Riemann tensor, for dimensional reasons in the system $\{y\}$ one has

$$\tilde{U}(y) = \sum_{k=0}^{p} A_k \left[\frac{d\tilde{\alpha}(\tau)}{d\tau} \right]^{2(p-k)} \left[\frac{d^2\tilde{\alpha}(\tau)}{d\tau^2} \right]^k, \tag{A.2}$$

and since under diffeomorphism U is a scalar quantity we also have

$$U(x) = \tilde{U}(y) = e^{-2pn(t)} \sum_{k=0}^{p} A_k \left[\frac{d\alpha(t)}{dt} \right]^{2(p-k)} \left[\frac{d^2\alpha(t)}{dt^2} - \frac{d\alpha(t)}{dt} \frac{dn(t)}{dt} \right]^k. \tag{A.3}$$

This means that in a generic system of coordinates an arbitrary invariant can be written in the form

$$U = e^{-2pn} H^{2p} \sum_{k=0}^{p} A_k X^k, \qquad X = \frac{\ddot{\alpha}}{\dot{\alpha}^2} - \frac{\dot{n}}{\dot{\alpha}} = \frac{\dot{H}}{H^2} - \frac{\dot{n}}{H}. \tag{A.4}$$

From the latter equation we directly get

$$\begin{aligned} U_{\dot{n}} &= -\frac{U_X}{H}, & U_n &= -2pU, \\ U_{\dot{H}} &= \frac{U_X}{H^2}, & U_H &= \frac{2pU}{H} - \frac{2\dot{H}U_X}{H^3} + \frac{\dot{n}U_X}{H^2} \end{aligned} \tag{A.5}$$

and for a generic function $F(U_1, U_2, U_3, \ldots, U_a, \ldots)$

$$\begin{aligned} F_{\dot{n}} &= -\frac{F_X}{H}, & F_n &= -2\sum_a p_a U_a F_{U_a}, \\ F_{\dot{H}} &= \frac{F_X}{H^2}, & F_H &= \frac{2}{H}\sum_a p_a U_a F_{U_a} - \frac{2X F_X}{H} - \frac{\dot{n}F_X}{H^2}. \end{aligned} \tag{A.6}$$

From equations above it directly follows

$$F_n + H F_H + 2\dot{H} F_{\dot{H}} + \dot{n} F_{\dot{n}} = 0. \tag{A.7}$$

After these considerations it is quite easy to recover (2.8). In fact, using (2.6), (2.7), (A.6)–(A.7) we get

$$\dot{E}_n - \dot{n} E_n - \dot{\alpha} E_\alpha = \frac{e^{n+3\alpha}}{2}[(\dot{n} + 3\dot{\alpha})(F_n + H F_H + 2\dot{H} F_{\dot{H}} + \dot{n} F_n)$$

$$+ \frac{d}{dt}(F_n + H F_H + 2\dot{H} F_{\dot{H}} + \dot{n} F_n)] = 0. \tag{A.8}$$

Before to end the appendix we also derive the useful relation

$$\left[\frac{F_H}{HF_{\dot{H}}}\right]_{(X=0,n=0)} = 4 \quad \Longrightarrow \quad \left[H\partial_H \log \frac{F_H}{F_{\dot{H}}}\right]_{(X=0,n=0)} = 1. \tag{A.9}$$

The latter identity can be derived as follows. From (A.6) we have

$$\left[\frac{F_H}{HF_{\dot{H}}}\right]_{(X=0,n=0)} = \left[\frac{HF_H}{F_X}\right]_{(X=0,n=0)} = \frac{2\sum_a p_a A_0(U_a) H^{2p_a} F_{U_a}}{\sum_a A_1(U_a) H^{2p_a} F_{U_a}}, \tag{A.10}$$

where $A_k(U_a)$ are the coefficients of the invariant U_a as in (A.4). Then (A.9) is true if for any invariant U_a one has

$$2A_1(U_a) = p_a A_0(U_a). \tag{A.11}$$

Such a relation is a direct consequence of the form of the Riemann tensor. In fact, in this case the non vanishing components read

$$R_{0a0a} = e^{2\alpha} H^2 (1+X), \qquad R_{abab} = -e^{-2n} e^{4\alpha} H^2, \quad a, b = 1, 2, 3$$

and since any invariant is built up with Riemann tensor it is easy to see that the relation (A.11) is always satisfied.

References

1. Sahni, V., Starobinsky, A.A.: Int. J. Mod. Phys. D **9**, 373 (2000)
2. Carroll, S.M.: Living Rev. **4**, 1 (2001)
3. Peeble, P.J.E., Ratra, B.: Rev. Mod. Phys. **75**, 559 (2003)
4. Padmanabhan, T.: Phys. Rept. **380**, 235 (2003)
5. AIP Conf. Proc. **861**, 179 (2006). arXiv:astro-ph/0603114
6. Copeland, E., Sami, M., Tsujikawa, S.: Int. J. Mod. Phys. D **15**, 1753 (2006)
7. Nojiri, S., Odintsov, S.D.: Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007)
8. Nojiri, S., Odintsov, S.D.: Can $F(R)$-gravity be a viable model: the universal unification scenario for inflation, dark energy and dark matter. arXiv:0801.4843 [astro-ph] (2008).
9. Cognola, G., Gastaldi, M., Zerbini, S.: On the stability of a class of modified gravitational models. Int. J. Theor. Phys. **47**, 898 (2008). arXiv:gr-qc/0701138
10. Wainwright, J., Ellis, G.F.R.: Dynamical Systems in Cosmology. Cambridge University Press, Cambridge (1997)
11. Goliath, M., Ellis, G.F.R.: Phys. Rev. D **60**, 023502 (1999)
12. Amendola, L., Gannouji, R., Polarski, D., Tsujikawa, S.: Phys. Rev. D **75**, 083504 (2007)
13. Amendola, L., Tsujikawa, S.: Phantom crossing, equation-of-state singularities, and local gravity constraints in f(R) models. ArXiv:0705.0396 (2007)
14. Carloni, S., Troisi, A., Dunsby, P.K.S.: Some remarks on the dynamical systems approach to fourth order gravity. arXiv:0706.0452 (2007)
15. Goheer, N., Leach, J.A., Dunsby, P.K.S.: Class. Quantum Gravity **25**, 035013 (2008)
16. Fay, S., Nesseris, S., Perivolaropoulos, L.: Phys. Rev. D **76**, 063504 (2007)
17. Carroll, S.M., De Felice, A., Duvvuri, V., Easson, D.A., Trodden, M., Turner, M.S.: Phys. Rev. D **71**, 063513 (2005)
18. Carroll, S.M., Duvvuri, V., Trodden, M., Turner, M.S.: Phys. Rev. D **70**, 043528 (2004)
19. Capozziello, S., Carloni, S., Troisi, A.: Quintessence without scalar fields. ArXiv:astro-ph/0303041 (2003)
20. Capozziello, S., Cardone, V.F., Troisi, A.: Phys. Rev. D **71**, 043503 (2005)
21. Brookfield, A.W., van de Bruck, C., Hall, L.M.H.: Phys. Rev. D **74**, 064028 (2006)
22. Sotiriou, T.P.: Class. Quantum Gravity **23**, 5117 (2006)
23. Nojiri, S., Odintsov, S.D.: Phys. Rev. D **74**, 086005 (2006)

24. Poplawski, N.J.: Phys. Rev. D **74**, 084032 (2006)
25. Borowiec, A., Godlowski, W., Szydlowski, M.: Phys. Rev. D **74**, 043502 (2006)
26. Koivisto, T.: Phys. Rev. D **73**, 083517 (2006)
27. de la Cruz-Dombriz, A., Dobado, A.: Phys. Rev. D **74**, 087501 (2006)
28. Multamaki, T., Vilja, I.: Phys. Rev. D **74**, 064022 (2006)
29. Sotiriou, T.P.: Phys. Lett. B **645**, 389 (2007)
30. Sotiriou, T.P., Liberati, S.: Ann. Phys. **322**, 935 (2007)
31. Faraoni, V., Nadeau, S.: Phys. Rev. D **75**, 023501 (2007)
32. Huterer, D., Linder, E.V.: Phys. Rev. D **75**, 023519 (2007)
33. Fay, S., Tavakol, R., Tsujikawa, S.: Phys. Rev. D **75**, 063509 (2007)
34. Kainulainen, K., Piilonen, J., Reijonen, V., Sunhede, D.: Phys. Rev. D **76**, 024020 (2007)
35. De Felice, A., Hindmarsh, M.: J. Cosmol. Astropart. Phys. **0706**, 028 (2007)
36. Uddin, K., Lidsey, J.E., Tavakol, R.: Class. Quantum Gravity **24**, 3951 (2007)
37. de Souza, J.C.C., Faraoni, V.: Class. Quantum Gravity **24**, 3637 (2007)
38. Movahed, M.S., Baghram, S., Rahvar, S.: Phys. Rev. D **76**, 044008 (2007)
39. Kahya, E.O., Onemli, V.K.: Phys. Rev. D **76**, 043512 (2007)
40. Bertolami, O., Bohmer, C.G., Harko, T., Lobo, F.S.N.: Phys. Rev. D **75**, 104016 (2007)
41. Fairbairn, M., Rydbeck, S.: Expansion history and f(R) modified gravity. arXiv:astro-ph/0701900 (2007)
42. Zhang, P.J.: Phys. Rev. D **73**, 123504 (2006)
43. Bludman, S.: What drives our accelerating universe? arXiv:astro-ph/0702085 (2007)
44. Capozziello, S., Francaviglia, M.: Extended theories of gravity and their cosmological and astrophysical applications. arXiv:0706.1146 [astro-ph] (2007)
45. De Felice, A., Mukherjee, P., Wang, Y.: Observational bounds on modified gravity models. arXiv:0706.1197 [astro-ph] (2007)
46. Bamba, K., Guo, Z.K., Ohta, N.: Accelerating cosmologies in the Einstein-Gauss-Bonnet theory with Dilaton. arXiv:0707.4334 [hep-th] (2007)
47. Boehmer, C.G., Harko, T., Lobo, F.: Dark matter as a geometric effect in f(R) gravity. arXiv:0709.0046 [gr-qc] (2007)
48. Hu, W., Sawicki, I.: Phys. Rev. D **76**, 064004 (2007)
49. Starobinsky, A.A.: JETP Lett. **86**, 157 (2007)
50. Nojiri, S., Odintsov, S.D.: Unifying inflation with LambdaCDM epoch in modified f(R) gravity consistent with Solar System tests. arXiv:0707.1941 [hep-th] (2007)
51. Appleby, S.A., Battye, R.A.: arXiv:0705.3199 [astro-ph] (2007)
52. Pogosian, L., Silvestri, A.: arXiv:0709.0296 [astro-ph] (2007)
53. Tsujikawa, S.: arXiv:0709.1391 [astro-ph] (2007)
54. Capozziello, S., Tsujikawa, S.: arXiv:0712.2268 [gr-qc] (2007)
55. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Sebastiani, L., Zerbini, S.: Phys. Rev. D **77**, 046009 (2008). arXiv:0712.4017 [gr-qc]
56. Nojiri, S., Odintsov, S.D.: Can F(R)-gravity be a viable model: the universal unification scenario for inflation, dark energy and dark matter. arXiv:0801.4843 [astro-ph] (2008)
57. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Zerbini, S.: J. Cosmol. Astropart. Phys. **0502**, 010 (2005)
58. Faraoni, V.: Ann. Phys. **317**, 366 (2005)
59. Faraoni, V.: Phys. Rev. D **72**, 061501 (2005) (R)
60. Faraoni, V.: Phys. Rev. D **75**, 067302 (2007)
61. Cognola, G., Zerbini, S.: J. Phys. A **39**, 6245 (2006)
62. Nojiri, S., Odintsov, S.D.: Phys. Rev. D **68**, 123512 (2003)
63. Capozziello, S., Nojiri, S., Odintsov, S.D., Troisi, A.: Phys. Lett. B **639**, 135 (2006)
64. Nojiri, S., Odintsov, S.D.: Phys. Rev. D **74**, 086009 (2006)
65. Nojiri, S., Odintsov, S.D.: J. Phys. Conf. Ser. **66**, 012005 (2007)
66. Bazeia, D., Carneiro da Cunha, B., Menezes, R., Petrov, A.Y.: Phys. Lett. B **649**, 445 (2007)
67. Rador, T.: Phys. Rev. D **75**, 064033 (2007)
68. Rador, T.: Phys. Lett. B **652**, 228 (2007)
69. Sokolowski, L.M.: Class. Quantum Gravity **24**, 3713 (2007)
70. Sokolowski, L.M.: Class. Quantum Gravity **24**, 3391–3411 (2007)
71. Bohmer, C.G., Hollenstein, L., Lobo, F.S.N.: Phys. Rev. D **76**, 084005 (2007)
72. Nojiri, S., Odintsov, S.D., Sasaki, M.: Phys. Rev. D **71**, 123509 (2005)
73. Antoniadis, I., Rizos, J., Tamvakis, K.: Nucl. Phys. B **415**, 497 (1994)
74. Mavromatos, N.E., Rizos, J.: Phys. Rev. D **62**, 124004 (2000)
75. Mavromatos, N.E., Rizos, J.: Int. J. Mod. Phys. A **18**, 57 (2003)

76. Nojiri, S., Odintsov, S.D.: Phys. Lett. B **631**, 1 (2005)
77. Sami, M., Toporensky, A., Tretjakov, P.V., Tsujikawa, S.: Phys. Lett. B **619**, 193 (2005)
78. Tsujikawa, S., Sami, M.: J. Cosmol. Astropart. Phys. **0701**, 006 (2007)
79. Koivisto, T., Mota, D.F.: Phys. Lett. B **644**, 104 (2007)
80. Koivisto, T., Mota, D.F.: Phys. Rev. D **75**, 023518 (2007)
81. Guo, Z.K., Ohta, N., Tsujikawa, S.: Phys. Rev. D **75**, 023520 (2007)
82. Calcagni, G., Carlos, B., De Felice, A.: Nucl. Phys. B **752**, 404 (2006)
83. Calcagni, G., Tsujikawa, S., Sami, M.: Class. Quantum Gravity **22**, 3977 (2005)
84. Sanyal, A.: Phys. Lett. B **645**, 1 (2007)
85. Neupane, I.P.: Class. Quantum Gravity **23**, 7493 (2006)
86. Carter, B.M.N., Neupane, I.P.: J. Cosmol. Astropart. Phys. **0606**, 004 (2006)
87. Nojiri, S., Odintsov, S.D., Gorbunova, O.G.: J. Phys. A **39**, 6627 (2006)
88. Brevik, I., Quiroga, J.: Int. J. Mod. Phys. D **16**, 817 (2007)
89. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Zerbini, S.: Phys. Rev. D **75**, 086002 (2007)
90. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Zerbini, S.: Phys. Rev. D **73**, 084007 (2006)
91. Nojiri, S., Odintsov, S.D., Sami, M.: Phys. Rev. D **74**, 046004 (2006)
92. Elizalde, E., Jhingan, S., Nojiri, S., Odintsov, S.D., Sami, M., Thongkool, I.: Dark energy generated from a (super)string effective action with higher order curvature corrections and a dynamical dilaton. arXiv:0705.1211 [hep-th] (2007)
93. Nojiri, S., Odintsov, S.D., Tretyakov, P.V.: Phys. Lett. B **651**, 224 (2007)
94. Vilenkin, A.: Phys. Rev. D **32**, 2511 (1985)
95. Capozziello, S.: Int. J. Mod. Phys. D **11**, 4483 (2002)
96. Cognola, G., Zerbini, S.: Vestn. TSPU **7**, 69 (2004)
97. Starobinsky, A.: Phys. Lett. B **91**, 99 (1980)
98. Starobinsky, A., Schmidt, H.J.: Class. Quantum Gravity **4**, 695 (1987)
99. Muller, V., Schmidt, H.J., Starobinsky, A.: Phys. Lett. B **202**, 198 (1988)
100. Schmidt, H.J.: Class. Quantum Gravity **5**, 233 (1988)
101. Navarro, I., Van Acoleyen, K.: J. Cosmol. Astropart. Phys. **0603**, 008 (2006)
102. Chiba, T.: J. Cosmol. Astropart. Phys. **0503**, 008 (2005)
103. Nunez, A., Solganik, S.: Phys. Lett. B **608**, 189 (2005)
104. Barrow, J.D., Hervik, S.: Phys. Rev. D **74**, 124017 (2006)
105. Li, B., Barrow, J.D., Mota, D.F.: Phys. Rev. D **76**, 044027 (2007)
106. Li, B., Barrow, J.D., Mota, D.F.: Phys. Rev. D 76, 104047 (2007)
107. Toporensky, A.V., Tretyakov, P.V.: Int. J. Mod. Phys. D **16**, 10751086 (2007)
108. Hindawi, A., Ovrut, B.A., Waldram, D.: Phys. Rev. D **53**, 5597 (1996)
109. Mena, O., Santiago, J., Weller, J.: Phys. Rev. Lett. **96**, 041103 (2006). arXiv:astro-ph/0510453
110. Goroff, M.H., Sagnotti, A.: Nucl. Phys. B **266**, 709 (1986)
111. van den Ven, A.E.M.: Nucl. Phys. B **378**, 309 (1992)